Perron-Frobenius Theorems

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This is a lecture note for Marxian Economic Thoery, a course at Renmin University of China.

In this note, we will discuss the *Perron-Frobenius Theorem*, which is one of the most powerful tools on nonnegative/positive matrices and the workhorse in mathematical Marxian economics. I assume that the reader are familiar with basic linear algebra.

This Note is written in Pluto Notebook, a reactive notebook for Julia.

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Nonnegative and Positive Matrices

Notations

In this note we use the following notations:

- A matrix is *positive*, $A > 0$, if and only if $a_{ij} > 0$ for all i, j.
- A matrix is *semi-positive*, $A \geq 0$, if and only if $a_{ij} \geqq 0$ for all i, j and $A \neq 0$.
- A matrix $A \geqq 0$ if it is *nonegative*.

As a vector can be taken as a speical matrix ($1 \times k$ or $k \times 1$ matrix), the above notations apply to vectors.

About the dimensions: By default, a matrix is $n \times n$ and vector $n \times n$ or determined by the context. For example, the vector v in vA is by default a row vector, while x is a column vector in Ax.

Example 1: Linear economy (A, ℓ)

Nonnegative matrices arise in many fields. In *Marxian Economic Theory*, it is used in the model of *linear economy* $\mathcal{E}(A, \ell)$.

Assume that there are n goods in the economy, and each column of the matrix A represents a production process.

$$a^j \oplus \ell_j \mapsto e_j$$

Specifically, a_{ij} is the amount of good i used in the production of 1-unit of good j. The row vector ℓ is the direct labor input, i.e., ℓ_j is the labor required in the production of 1-unit of good j.

The nonnegative matrix A is called an *input-output matrix*. Usually, we assume the input-output matrix A is *productive* and *indecomposable*, which will be discussed later.

Example 2: Markov Chain

Suppose that there are n states, and the probability of jumping from state j to state i is given by $p_{ij} \geqq 0$, then

$$\sum_{i=1}^{n} p_{ij} = 1, \text{ for all } j$$

The nonnegative matrix $P = (p_{ij})$ is called a *stochastic matrix*.

Indecomposable Matrices

A square matrix A is *decomposable* if it can be reorganized by performing the same permutation on the rows and the columns into the form of

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

where A_{11} and A_{22} are square.

For example, if one column of the matrix A is zero vector, say $a^1 = 0$, then A is decomposable.

An equivalent definition: if the set $\{1, \ldots, n\}$ can be partitioned into two disjoint subsets I and J such that $a_{ij} = 0$ for all $i \in I, j \in J$, then A is decomposable.

A square matrix A is said to be *indecomposable* if it is not decomposable. In the context of Markov chains, it is also called *irreducible*.

Note that if a nonnegative matrix A is indecomposable, then it must be semi-positve, $A \geq 0$.

If A is indecomposable, the so is the transpose A'.

Some Preliminary Properties

- If $A > 0$, $x \geq 0$, then $Ax > 0$.

- If $A \geqq 0$ is indecomposable, $x \geq 0$, then $Ax \geq 0$. For if $Ax = \sum_i^n x_i a^i = 0$, and $x_i > 0$, then $a^i = 0$ and A is decomposable.

Eigenvalues and Eigenvectors of Nonnegative Matrices

The Perron-Frobenius Theorem

Theorem

Let $A \geq 0$ be indecomposable, then

(1) A has a positive eigenvalue $\lambda(A) > 0$ associated with a positive eigenvector $v > 0$

$$Av = \lambda(A)v$$

(2) If $u \geq 0$ is an eigenvector, then it has eigenvalue $\lambda(A)$, and $u > 0$ is a multiple of v.

(3) If α is any eigenvalue of A, then $|\alpha| < \lambda(A)$.

(4) If $A \geq B \geqq 0$ then $\lambda(A) > \lambda(B)$. Moreover, if $A \geqq B \geqq 0$ and $\lambda(B) = \lambda(A)$, then $B = A$.

Statement (1) and (2) mean that A processes a *unique* nonnegative eigenvector (up to a scalar), and its associated eigenvalue is positive. We call this unique eigenvalue $\lambda(A)$ the **Frobenius root** of the nonnegative matrix A.

Then (3) means that the Frobenius root is the largest eigenvalue, and (4) implies that the Frobenius root is an *increasing* and *continuous* function of the matrix.

If $A \geqq 0$ without the assumption of indecomposability, then $\lambda(A) \geqq 0$ and $v \geq 0$. Similarly, (3) and (4) hold with weak inequality.

Moreover, if $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ where A_{11} is square. Consider $B = \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix}$ then

$$\lambda(A_{11}) = \lambda(B) < \lambda(A)$$

There are many different ways to prove this theorem. I present one of them using the fixed point theorem in the last section. Next let's first look at some examples.

```
A = 2×2 Matrix{Float64}:
     0.5  0.2
     0.1  0.6
```

```julia
A = [0.5 0.2
     0.1 0.6]
```

```
Frobenius (generic function with 1 method)
```

```
function Frobenius(A)
    # a function returing the Frobenius root and an associated positive eigenvector
    evals, evecs = eigen(A)
    root, arg = findmax(evals)
    vector = abs.(evecs[:,arg])
    return (root, vector)
end
```

```
(0.7, [0.707107, 0.707107])
```

```
λ_A, v_A =  Frobenius(A)
```

```
B = 2×2 Matrix{Float64}:
 0.4  0.2
 0.1  0.6
```

```
B = [0.4 0.2
    0.1 0.6]  ## B < A
```

```
(0.673205, [0.59069, 0.806898])
```

```
λ_B, v_B = Frobenius(B)  # lambda(B)  < lambda(A)
```

As A' has the same eigenvalues as A, we have $\lambda(A') = \lambda(A)$. Then

$$A'v = \lambda(A')v = \lambda(A)v$$

and therefore

$$v'A = \lambda(A)v'$$

```
(0.7, [0.447214, 0.894427])
```

```
λ_At, v_At =  Frobenius(A')  # labmda(A')= lambda(A)
```

```
1×2 adjoint(::Vector{Float64}) with eltype Float64:
 0.31305  0.626099
```

```
v_At' * A  #v'A
```

```
1×2 adjoint(::Vector{Float64}) with eltype Float64:
 0.31305  0.626099
```

```
λ_A * v_At'  # lambda(A) * v', the same result
```

Let A be nonnegative, indecomposable, and $\lambda(A)$ its Frobenius root, we have

- For an $x \geq 0$,

$$Ax \leqq sx \Rightarrow \lambda(A) \leqq s$$

$$Ax \geqq sx \Rightarrow \lambda(A) \geqq s$$

- For an $x \geq 0$,

$$Ax \leq sx \Rightarrow \lambda(A) < s$$

$$Ax \geq sx \Rightarrow \lambda(A) > s$$

The proofs of these four statements are identical. The readers are encouraged the prove them as an exercise. Below we present only the proof of the last one.

Hint

Let $v > 0$ be a eigenvector of the matrix A' associated with $\lambda(A') = \lambda(A)$

$$A'v = \lambda(A)v$$

then $v'x > 0$ with $x \geqq 0$. Since $Ax \geqq sx$, then

$$v'Ax > sv'x = \lambda(A)v'x > sv'x \Rightarrow \lambda(A) > s.$$

If A is nonnegative, without the assumption of indecomposability, then we have

- For an $x > 0$,

$$Ax \leqq sx \Rightarrow \lambda(A) \leqq s$$

$$Ax \geqq sx \Rightarrow \lambda(A) \geqq s$$

- For an $x \geqq 0$,

$$Ax < sx \Rightarrow \lambda(A) < s$$

$$Ax > sx \Rightarrow \lambda(A) > s$$

Max-min and Min-max Characterization

Let A be nonnegative, indecomposable and $\lambda(A)$ its Frobenius root. It is immediately from the above observations that

- if $w > 0$, $Aw \leqq \lambda(A)w$, then $Aw = \lambda(A)w$
- if $w > 0$, $Aw \geqq \lambda(A)w$, then $Aw = \lambda(A)w$

In other words, it is impossible to have

$$Aw \leq \lambda(A)w$$

or

$$Aw \geq \lambda(A)w$$

for any $w > 0$. Then, $\lambda(A)$ cannot be the minimum nor the maximum of the ratio $\frac{(Aw)_i}{w_i}$.

Therefore, we have the following theorem

Theorem

Let A be nonnegative, indecomposable and $\lambda(A)$ its Frobenius root. Let $w > 0$, then either

$$\min_i \frac{(Aw)_i}{w_i} < \lambda(A) < \max_i \frac{(Aw)_i}{w_i},$$

or

$$\min_i \frac{(Aw)_i}{w_i} = \lambda(A) = \max_i \frac{(Aw)_i}{w_i}.$$

```
w = ▸[0.2, 0.5]
```

For example, consider the $w > 0$ above, then

- the minimum of $\frac{(Aw)_i}{w_i}$ is 0.64;
- the maximum of $\frac{(Aw)_i}{w_i}$ is 1.0;
- while $\lambda(A) =$ 0.7 is in between.

```
▸[1.0, 0.64]
(A * w) ./ w  # (Aw)_i / w_i
```

```
min = 0.64
min = minimum(A * w ./ w)
```

```
max = 1.0
max = maximum(A * w ./w)
```

Productive Matrice

A nonnegative square matrix is said to be *productive* if there exists $x \geq 0$ such that

$$x - Ax \geq 0$$

Exercise.

Let $A \geq 0$ be indecomposable. Show that A is productive if and only if $\lambda(A) < 1$.

Hint

[illegible]

Theorem

Let $A \geq 0$ be indecomposable, then $s > \lambda(A)$ if and only if $(sI - A)^{-1} > 0$.

Proof

- ($\Rightarrow$) By the Perron-Frobenius Theorem, $\lambda(A)$ is the largest eigenvalue of A, i.e., the largest root of the equation

  $$\det(\lambda I - A) = 0$$

  Then $s > \lambda(A)$ is not a root, i.e., $\det(sI - A) \neq 0$. Therefore, $(sI - A)^{-1}$ exists.

  Next, we show that $(sI - A)^{-1} > 0$. It is sufficient to show that $z = (sI - A)^{-1} y > 0$ for any $y \geq 0$. Suppose that z has some negative components,

  $$z = \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix}$$

  where $x_1 > 0$ and $x_2 \geqq 0$, then

  $$\begin{bmatrix} sI - A_{11} & -A_{12} \\ -A_{21} & sI - A_{22} \end{bmatrix} \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix} = y$$

  implies $-(sI - A_{11})x_1 - A_{12}x_2 \geqq 0$. Then

  $$(sI - A_{11})x_1 \leqq -A_{12}x_2 \leqq 0 \Rightarrow sx_1 \leqq A_{11}x_1$$

  and therefore $\lambda(A_{11}) \geqq s$, a contradiction to the fact that $\lambda(A_{11}) \leqq \lambda(A) < s$. Therefore, we have $z \geqq 0$.

  Suppose that z has zero components, then

  $$\begin{bmatrix} sI - A_{11} & -A_{12} \\ -A_{21} & sI - A_{22} \end{bmatrix} \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = y$$

  where $x_2 > 0$. Then

  $$-A_{12}x_2 \geqq 0 \Rightarrow A_{12}x_2 = 0$$

  and then $A_{12} = 0$ since $x_2 > 0$, violating the indecomposability of A. Therefore, $z > 0$.

- ($\Leftarrow$) By $(sI - A)^{-1} > 0$, let $y > 0$ and $x = (sI - A)^{-1} y > 0$, then

  $$y = (sI - A)x = sx - Ax > 0$$

  That is, we have some $x > 0$ such that $sx > Ax$, then $s > \lambda(A)$.

Applying the above theorem to the case with $\lambda(A) < 1$ when A is productive, we have

Take-Home Message

If a nonnegative matrix A is indecomposable and productive, then $(I - A)^{-1} > 0$.

Moreover,

$$(I - A)^{-1} = I + A + A^2 + \cdots = \sum_{k=0}^{\infty} A^k$$

holds when $\lambda(A) < 1$, as a generalization of

$$\frac{1}{1-q} = 1 + q^2 + \cdots = \sum_{k=0}^{\infty} q^k, \quad \text{for } |q| < 1$$

Appendix: The proof of the Perron-Frobenius Theorem

Theorem

Let $A \geq 0$ be indecomposable, then

(1) A has a positive eigenvalue $\lambda(A) > 0$ associated with a positive eigenvector $v > 0$

$$Av = \lambda(A)v$$

(2) If $u \geq 0$ is an eigenvector, then it has eigenvalue $\lambda(A)$, and $u > 0$ is a multiple of v.

(3) If α is any eigenvalue of A, then $|\alpha| < \lambda(A)$.

(4) If $A \geq B \geqq 0$ then $\lambda(A) > \lambda(B)$. Moreover, if $A \geqq B \geqq 0$ and $\lambda(B) = \lambda(A)$, then $B = A$.

We first establish (1) using the following fixed point theorem.

Let $\Delta = \{x \geq 0 \mid x \in \mathbb{R}^n \text{ and } \|x\| = 1\}$, and $f : \Delta \to \Delta$ be a continous function. The **Brouwer fixed point theorem** ensures that f has a fixed point, i.e., there exists $x_0 \in \Delta$ such that $f(x_0) = x_0$.

For a nonnegative and indecomposable matrix A, we first show that there exist a positive eigenvalue $\lambda > 0$ such that

$$Av = \lambda v$$

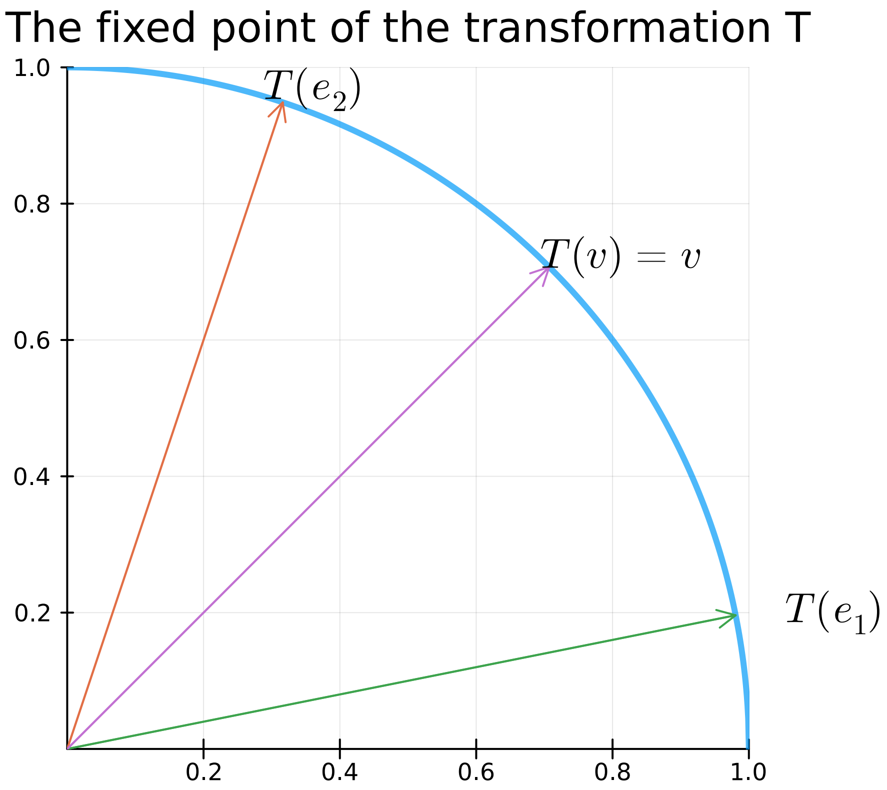
Since A is indecomposable, for any $x \geq 0$, we have $Ax \geq 0$, then $\|Ax\| > 0$. Define $T : \Delta \to \Delta$ by

$$T(x) = \frac{Ax}{\|Ax\|}, \forall x \in \Delta$$

then T is continous and there exists a fixed point $v \in \Delta$ such that

$$T(v) = \frac{Av}{\|Av\|} = v \Rightarrow Av = \lambda v$$

where $\lambda = \|Av\| > 0$.



⚠ lims should be a Tuple, not StepRangeLen{Float64, Base.TwicePrecision{Float64}, Base.TwicePrecision{Float64}, Int64}.

Next, we show that v must be positve by the proof of contradiction. Without loss of generality, suppose on the contrary that v has some zero components,

$$v = \begin{bmatrix} v_1 \\ 0 \end{bmatrix}$$

where $v_1 > 0$. Then $Av = \lambda v$ yields

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda v_1 \\ 0 \end{bmatrix}$$

Thus $A_{21} v_1 = 0$, so $A_{21} = 0$, violating the indecomposability of A.

For (2), let $w > 0$ be an eigenvector of A' associated with $\lambda(A') = \lambda(A)$. Suppose that $Au = \lambda_u u$, then

$$\lambda_u w'u = w'Au = \lambda(A)w'u$$

so $\lambda_u = \lambda(A)$ since $w'u > 0$. Then $u > 0$ by the same argument (showing v>0) above.

Suppose that u is not a multiple of v, then there exist $k \in \mathbb{R}$ such that $u + kv \geq 0$ has zero component. Note that $u + kv$ is an eigenvector associated with $\lambda(A)$, so $u + kv > 0$ contradicted.